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DETECTION OF SINGULARITIES USING SEGMENT APPROXIMATION

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ABSTRACT. We discuss best segment approximation (with free knots) by polynomials to piecewise analytic functions on a real interval. It is shown that, if the degree of the polynomials tends to infinity and the number of knots is the same as the number of singularities of the function, then the optimal knots converge geometrically fast to the singularities. When the degree is held fixed and the number of knots tends to infinity, we study the asymptotic distribution of the optimal knots.

1. INTRODUCTION AND MAIN RESULTS

The notion of segment approximation can be described in a very general way as follows. Let L be a functional that associates with each continuous $f: [-1, 1] \to \mathbb{R}$ and each subinterval $[a, b] \subseteq [-1, 1]$ a nonnegative number $L(f, [a, b])$. Usually L describes some constructive property of f on $[a, b]$ such as the modulus of continuity of f on $[a, b]$, the degree of approximation of f by polynomials of a fixed degree on $[a, b]$, etc.

A partition

$$-1 = x_0 < x_1 < \cdots < x_{k+1} = 1 \tag{1.1}$$

is called a *leveled partition* if

$$L(f, [x_i, x_{i+1}]) = L(f, [x_j, x_{j+1}]), \qquad 0 \le i, j \le k,$$

and an *optimal partition* if

$$\max_{0\le i\le k} L(f, [x_i, x_{i+1}]) \le \max_{0\le i\le k} L(f, [z_i, z_{i+1}])$$

for all partitions $-1 = z_0 < z_1 < \cdots < z_{k+1} = 1$. The problem of segment approximation consists of determining an optimal partition $\{x_i\}$ and the value

$$\max_{0\le i\le k} L(f, [x_i, x_{i+1}])$$

for this partition. It is easy to see that all leveled partitions are optimal if L is monotone on the intervals, i.e.,

$$L(f, I_1) \le L(f, I_2) \quad \text{for all intervals } I_1 \subseteq I_2 \subseteq [-1, 1].$$

In [4, 5], one can find efficient algorithms to determine the leveled and optimal partitions for such functionals. However, the existence of a leveled partition is guaranteed by the following theorem for a fairly large class of functionals L.

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Theorem 1.1. *Let* $L: [-1, 1] \times [-1, 1] \to [0, \infty)$ *be continuous and* $L(a, a) = 0$ *for* $a \in [-1, 1]$. *Then for each* $k \geq 1$ *there exists a partition* $-1 = x_0 < x_1 < \cdots < x_{k+1} = 1$ *such that*

$$L(x_i, x_{i+1}) = L(x_j, x_{j+1}), \qquad 0 \leq i, j \leq k.$$

In this paper, we are mainly interested in the following functionals. For an integer $n \geq 0$, Π_n will denote the class of all real polynomials of degree at most n. For $n \geq 0$ we define the functional d_n as follows. If $[a, b] \subseteq \mathbb{R}$ and $f \in C[a, b]$, then

$$d_n(f, [a, b]) := \min_{p \in \Pi_n} \|f - p\|_{[a, b]}, \tag{1.2}$$

where $\|\cdot\|$ denotes the supremum norm on $[a, b]$. Next, for $n \geq 1$ and $[a, b] \subseteq \mathbb{R}$, let

$$t_{j,n} := t_{j,n,a,b} := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j-1}{2n}\pi\right), \qquad j = 1, \ldots, n.$$

For $f \in C[a, b]$, let $L_n(f, [a, b]) \in \Pi_{n-1}$ be the polynomial satisfying

$$L_n(f, [a, b])(t_{j,n}) = f(t_{j,n}), \qquad 1 \leq j \leq n. \tag{1.3}$$

We define the functional $\tilde{d}_n$ by

$$\tilde{d}_n(f, [a, b]) := \|f - L_n(f, [a, b])\|_{[a, b]}. \tag{1.4}$$

Of course, $\tilde{d}_n$ is much easier to compute than d_n. Furthermore, practical experience shows that the algorithm in [5] converges to a leveled partition for the functionals $\tilde{d}_n$ in (1.4).

The following theorem demonstrates that the leveled partitions corresponding to the functionals (1.2) and (1.4) can be used to detect certain singularities of f.

Theorem 1.2. *Let* $f \in C[a, b]$ *be piecewise analytic, i.e., there is a partition*

$$-1 = y_0 < y_1 < \cdots < y_{k+1} = 1,$$

such that for each integer i, $0 \leq i \leq k$, $f_i := f|_{[y_i, y_{i+1}]}$ *has an analytic continuation to a neighborhood of* $[y_i, y_{i+1}]$, *but does not extend to an analytic function on any neighborhood of any* y_i, $i = 1, \ldots, k$. *Let, for each integer* $n \geq 1$,

$$-1 = x_{n,0} < x_{n,1} < \cdots < x_{n,k+1} = 1$$

be a leveled partition for f *corresponding to* d_n *(respectively* $\tilde{d}_n$*). Then, for each integer* i, $1 \leq i \leq k$, $\lim_{n\to\infty} x_{n,i} = y_i$. *In fact,*

$$\limsup_{n\to\infty} \sup_{1 \leq i \leq k} |x_{n,i} - y_i|^{1/n} < 1. \tag{1.5}$$

The proof of Theorem 1.2 depends on the characterization of analytic functions due to Bernstein and a theorem of Hasson on derivatives of polynomials of best approximation. When more information is available about the function, for example when the functions f_i are entire functions of given order and type, the rates of convergence in (1.5) can be strengthened accordingly (see Corollary 3.1).

The following counterexample presents a case where the internal knot does not converge to the internal singularity of the function f. However, in this case the function is not differentiable in the first segment interval.

Example 1.3. Let $k = 1$ and $-1 < x_n := x_{1,n} < 1$ be a leveled partition for the following function f corresponding to the functional d_n:

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } -1 \le x \le 0, \\ 1 + x/2 & \text{for } 0 < x \le 1. \end{cases}$$

Then we claim that x_n converges to -1. Assume that $x_n > -1+\varepsilon$ for $n \in \Lambda$, where Λ is a subsequence of $\mathbb{N}$ and $\varepsilon > 0$. By a theorem of Bernstein [1], for $\varepsilon > 0$,

$$\lim_{n\to\infty,\, n\in\Lambda} n d_n(f, [-1, -1+\varepsilon]) = C\sqrt{\varepsilon}.$$

Thus,

$$\limsup_{n\to\infty} n d_n(f, [-1, x_n]) \ge C_1 > 0.$$

On the other hand, f is continuously differentiable on $[-1+\varepsilon, 1]$, and f' satisfies a Lipschitz condition of order 1. By a theorem of Jackson [6],

$$\limsup_{n\to\infty} n d_n(f, [x_n, 1]) = 0.$$

Since, by the definition of x_n, $d_n(f, [-1, x_n]) = d(f, [x_n, 1])$ for all integers $n \ge 1$, this is a contradiction.

Next we consider the case when the degree of polynomials stays fixed, but the number of knots tends to infinity. For any partition

$$P\colon\ -1 = z_0 < z_1 < \cdots < z_{k+1} = 1,$$

we define the measure τ_P by the formula

$$\tau_P(A) := \frac{|\{j\colon z_j \in A\}|}{k+2}$$

for Borel sets $A \subseteq [-1, 1]$.

Theorem 1.4. *Let $n \ge 0$ be a fixed integer, f be $n+1$ times continuously differentiable on $[-1, 1]$, and for an integer $k \ge 0$*

$$P_k\colon\ -1 = \xi_{k,0} < \xi_{k,1} < \cdots < \xi_{k,k+1} = 1$$

be a leveled partition for f corresponding to d_n (respectively $\tilde{d}_n$). Furthermore, we assume that f is not a polynomial in Π_n on any subinterval $[-1, 1]$. Set $\lambda := \int_{-1}^{1} |f^{(n+1)}(t)|^{1/n}\,dt$ and $E_k := d_n(f, [\xi_{k,0}, \xi_{k,1}])$ (respectively $E_k := \tilde{d}_n(f, [\xi_{k,0}, \xi_{k,1}])$). Then

$$\lim_{k\to\infty} (k+1)[(n+1)!2^n E_k]^{1/n} = \lambda$$

and, for any $[a, b] \subseteq [-1, 1]$,

$$\lim_{k\to\infty} \tau_{P_k}([a, b]) = \frac{1}{\lambda}\int_a^b |f^{(n+1)}(t)|^{1/n}\,dt.$$

2. Numerical results

In our first example, we set

$$f_1(x) := |\cos\pi(x - 0.1)|, \qquad x \in [-1, 1],$$

TABLE 1

degree	y_0	y_1	y_2	y_3	error
3	-1	-0.3907	0.5899	1	0.026
5	-1	-0.3998	0.5992	1	0.00059
7	-1	-0.3999978	0.6000002	1	0.0000068

TABLE 2

degree	y_0	y_1	y_2	error
3	0	0.057	1	0.01
5	0	0.02	1	0.0039
7	0	0.015	1	0.0025

TABLE 3

degree	y_0	y_1	y_2	error
3	0.1	0.4035	1	0.00083
5	0.1	0.2538	1	0.00000815
7	0.1	0.2498	1	0.00000007

so that the true singularities y_1 and y_2 are respectively -0.4 and 0.6. The function f_1 is approximated with the help of an algorithm based on [5]. The results are summarized in Table 1, where *error* means deviation of f_1 from its best segment approximation.

Next we choose the function

$$f_2(x) := \begin{cases} \sqrt{x}, & 0 \le x \le 1/4, \\ 1/2e^{2x-1/2}, & 1/4 \le x \le 1. \end{cases}$$

First we approximate this function on $[0, 1]$ with one internal knot (Table 2). As expected from Example 1.3 we have $y_1 \to 0$.

Finally, we approximate f_2 on the interval $[0.1, 1]$ with one internal knot (Table 3).

3. PROOFS

Proof of Theorem 1.1. We may assume that for each partition (1.1) there is at least one $j \in \{1, \dots, k+1\}$ with $L(x_{j-1}, x_j) > 0$. For, otherwise, the theorem is already proved. Define

$$M = \left\{ (\delta_1, \dots, \delta_{k+1}) \in \mathbb{R}^{k+1} \colon \sum_{i=1}^{k+1} \delta_i = 2, \ \delta_i \geq 0 \text{ for } i = 1, \dots, k+1 \right\}$$

and set

$$S(d) := 2 \frac{(p_1, \dots, p_{k+1})}{\sum_{i=1}^{k+1} p_i}$$

for $d = (\delta_1, \dots, \delta_{k+1}) \in M$, where

$$p_i := L\left(-1 + \sum_{j=1}^{i-1} \delta_j, \ -1 + \sum_{j=1}^{i} \delta_j \right).$$

Obviously, S is a continuous operator, mapping M to itself. It suffices to prove that there is a $d \in M$ with

$$S(d) = \zeta := \left(\frac{2}{k+1}, \dots, \frac{2}{k+1} \right). \tag{3.1}$$

Assume the contrary. Then we define an operator $T \colon M \to M$ as follows. For $m \in M$ we construct the straight line L_m which runs through $S(m)$ and ζ. The intersection of L_m with M is a line segment with endpoints p_1 and p_2. Set $T(m) = p_1$ if p_1 is the endpoint such that ζ lies between p_1 and $S(m)$, and p_2 otherwise. Since $S(m) \neq \zeta$, the operator T is well defined and continuous. Furthermore, T maps M to itself.

In view of Brower's fixed point theorem, T has a fixed point $d = (\delta_1, \dots, \delta_{k+1}) \in M$. By the construction of T, $\delta_j = 0$ for a $j \in \{1, \dots, k+1\}$. Let $S(d) =: (s_1, \dots, s_{k+1})$. Since $L(a, a) = 0$ for all $a \in [-1, 1]$, we see that $s_j = 0$. The construction of T then implies that the jth component of $T(d) = d$ is not zero, i.e., $\delta_j \neq 0$. This contradiction proves that (3.1) holds for some $d \in M$. □

Proof of Theorem 1.2. We observe that a piecewise analytic function cannot be infinitely many times differentiable on $[-1, 1]$. For $1 \leq i \leq k$, let $r_i > 0$ be the smallest integer such that $f^{(r_i)}(y_i+) \neq f^{(r_i)}(y_i-)$ and set

$$\sigma := \inf_{1 \leq i \leq k} |f^{(r_i)}(y_i+) - f^{(r_i)}(y_i-)| > 0.$$

We now fix i, $1 \leq i \leq k$. For integers $n \geq 0$, we set $\varepsilon_{i,n} := \inf_{1 \leq l \leq k} |x_{n,l} - y_i|$, and we shall estimate σ in terms of $\varepsilon_{i,n}$. For the sake of concreteness, we shall assume that $\{x_{n,l}\}$ is a leveled partition for $\tilde{d}_n$. The proof for the case of d_n is similar and simpler.

Let $a_j < y_j < y_{j+1} < b_j$ be found so that f_j can be extended to an analytic function on a neighborhood of $[a_j, b_j]$, $j = 0, \dots, k$. Let

$$D_n := \max_{0 \leq j \leq k} d_n(f_j, [a_j, b_j])$$

and, for any j, $0 \leq j \leq k$,

$$\widetilde{D}_n := \tilde{d}_n(f, [x_{n,j}, x_{n,j+1}]).$$

We observe that at least one of the intervals $[x_{n,j}, x_{n,j+1}]$ is contained in an interval $[y_l, y_{l+1}]$, $0 \le j, l \le k$. Thus, it follows from a well-known result [3, Vol. III, p. 48] that

$$\widetilde{D}_n \le c \log n D_n . \tag{3.2}$$

(Here and in the sequel, $c, c_1, \dots$ will denote positive constants independent of n and the function involved.)

Let

$$P_n(t) := \begin{cases} L_n(f, [x_{n,j}, x_{n,j+1}])(t), & x_{n,j} \le t < x_{n,j+1},\ 0 \le j \le k-1, \\ L_n(f, [x_{n,k}, 1])(t), & x_{n,k} \le t \le 1. \end{cases}$$

Then, by (3.2),

$$\|f - P_n\|_{[-1,1]} \le c \log n D_n . \tag{3.3}$$

Let $n \in \mathbb{N}$ such that $\varepsilon_{i,n} > 0$. If $p_n \in \Pi_n$ denotes the polynomial of best approximation to f_i on $[a_i, b_i]$, then by a theorem due to Hasson [2], we have

$$\|p_n^{(r_i)} - f_i^{(r_i)}\|_{[y_i, y_{i+1}]} \le c_1 n^r D_n . \tag{3.4}$$

Let $K_i := [y_i, y_i + \varepsilon_{i,n}]$. Then P_n is a polynomial on K_i. Using the Markov inequality and (3.3), we obtain

$$\|P_n^{(r_i)} - p_n^{(r_i)}\|_{K_i} \le \frac{c_2 n^{2r_i}}{\varepsilon_{i,n}^{r_i}} \|P_n - p_n\|_{K_i} \le \frac{c_3 n^{2r_i} \log n D_n}{\varepsilon_{i,n}^{r_i}} .$$

Hence, (3.4) implies

$$|P_n^{(r_i)}(y_i) - f^{(r_i)}(y_i+)| = |P_n^{(r_i)}(y_i) - f_i^{(r_i)}(y_i)| \le \frac{c_4 n^{2r_i} \log n D_n}{\varepsilon_{i,n}^{r_i}} .$$

Estimating $|P_n^{(r_i)}(y_i) - f^{(r_i)}(y_i-)|$ in the same manner, we see that

$$\sigma \le |f^{(r_i)}(y_i+) - f^{(r_i)}(y_i-)| \le c_5 \frac{n^{2r_i} \log n D_n}{\varepsilon_{i,n}^{r_i}} .$$

Hence,

$$\varepsilon_{i,n} \le c_6 \frac{n^2 (\log n D_n)^{1/r_i}}{\sigma^{1/r_i}} . \tag{3.5}$$

This estimate is valid for all $n \in \mathbb{N}$ and $1 \le i \le k$. Since $D_n \to 0$ exponentially fast, $\varepsilon_{i,n} \to 0$ exponentially fast, $1 \le i \le k$. However, this is only possible if $x_{n,i} \to y_i$ exponentially fast. This proves Theorem 1.2. □

It is clear from the proof that the operator L_n can be replaced by a wide class of operators. As long as $\log n$ in (3.3) is replaced by constants C_n such that $\limsup_{n\to\infty} C_n^{1/n} = 1$, the theorem remains true.

Furthermore, we note the following corollaries of the proof of Theorem 1.2.

Corollary 3.1. *If each f_i is an entire function of order 1 and finite exponential type τ, $0 \le i \le k$, then for any $\delta > 0$, using the notation of the proof of Theorem 1.2,*

$$\varepsilon_{i,n} \le \frac{c(\tau + \delta)^n}{\sigma^{1/r_i} n!} .$$

Corollary 3.2. *Let $m \geq 2$ and $k \geq 1$ be integers, $\{y_i\}_{i=0}^{k+1}$ be as in Theorem 1.2, and $f\colon [-1, 1] \to \mathbb{R}$ be m-times continuously differentiable on each interval $[y_i, y_{i+1}]$, $0 \leq i \leq k$, but not $([m/2]-1)$-times continuously differentiable in the points y_i, $i = 1, \ldots, k$. Let $\{x_{n,i}\}_{i=0}^{k+1}$ be the leveled partitions for f as in Theorem 1.2. Then $\lim_{n\to\infty} x_{n,i} = y_i$ for $1 \leq i \leq k$.*

Proof of Theorem 1.4. As in the proof of Theorem 1.2, we consider only the case of the functional d_n: the case of $\tilde{d}_n$ is simpler. Denote by E_k the error of approximation with k internal knots. Since

$$\limsup_{k\to\infty} \sup_{0\leq\eta\leq 2/(k+1),\, -1\leq x\leq 1-\eta} d_n(f, [x, x+\eta]) = 0$$

and at least one of the intervals $[\xi_{k,j}, \xi_{k,j+1}]$ has length not exceeding $2/(k+1)$, E_k tends to 0 as $k \to \infty$. Set $\varepsilon_{k,\nu} := \xi_{k,\nu+1} - \xi_{k,\nu}$, $\nu = 0, \ldots, k$.

We claim that

$$\lim_{k\to\infty} \max_{0\leq\nu\leq k} \varepsilon_{k,\nu} = 0. \tag{3.6}$$

If not, a compactness argument shows that there is a subinterval $I = [\alpha, \beta]$, $\alpha < \beta$, of $[-1, 1]$ such that

$$I \subseteq [\xi_{k,\nu(k)}, \xi_{k,\nu(k)+1}], \qquad k = k_1, k_2, \ldots,$$

for a subsequence $k_1 < k_2 < \cdots$ of $\mathbb{N}$. Since E_k tends to 0, the function f must be a polynomial of degree n on I. This contradiction proves our claim.

Let

$$I_{k,\nu} := [\xi_{k,\nu}, \xi_{k,\nu+1}], \qquad k \in \mathbb{N},\ \nu = 0, \ldots, k,$$

and define $p_{k,\nu} \in \Pi_n$ to be the polynomial of best approximation to f on $I_{k,\nu}$. Interpolating in the roots of the nth-degree Chebyshev polynomial scaled to the interval I_ν, a well-known formula for the error of interpolation [3, Vol. III, p. 10] implies that

$$E_k \leq \max_{I_\nu} \frac{|f^{(n+1)}|}{(n+1)!} \frac{\varepsilon_{k,\nu}^n}{2^{n-1}}. \tag{3.7}$$

On the other hand, in view of the Chebyshev alternation theorem, $p_{k,\nu}$ interpolates f on at least n distinct points $\tilde{t}_{k,\nu,1} < \cdots < \tilde{t}_{k,\nu,n}$ in I_ν. Hence,

$$f(x) - p_{k,\nu}(x) = \frac{f^{n+1}(\xi_x)}{(n+1)!} \omega_{k,\nu}(x),$$

where ξ_x is a point in $I_{k,\nu}$ and $\omega_{k,\nu}(x) = \prod_{\mu=1}^n (x - \tilde{t}_{k,\nu,\mu})$. Since the Chebyshev polynomial (scaled to $I_{k,\nu}$) has the minimal norm among all monic polynomials in Π_n, we have

$$\max_{t\in I_{k,\nu}} |\omega_{k,\nu}(t)| =: |\omega(s_{k,\nu})| \geq \frac{\varepsilon_{k,\nu}^n}{2^{n-1}}. \tag{3.8}$$

Hence,

$$\begin{aligned} E_k &\geq |(f - p_{k,\nu})(s_{k,\nu})| \geq \frac{\min_{t\in I_{k,\nu}} |f^{(n+1)}|}{(n+1)!} \max_{t\in I_{k,\nu}} |\omega_{k,\nu}(t)| \\ &\geq \frac{\min_{t\in I_{k,\nu}} |f^{(n+1)}|}{(n+1)!} \frac{\varepsilon_{k,\nu}^n}{2^{n-1}}. \end{aligned}$$

Setting $c := ((n+1)!2^{n-1})^{1/n}$, we get

$$\min_{t\in I_\nu} |f^{(n+1)}(t)|^{1/n}\varepsilon_{k,\nu} \le cE_k^{1/n} \le \max_{t\in I_\nu} |f^{(n+1)}(t)|^{1/n}\varepsilon_{k,\nu}\,.$$

By summing over $\nu = 0, \dots, k$, it follows from the Riemann integrability of $|f^{(n+1)}|^{1/n}$ that

$$\lim_{k\to\infty} (k+1)cE_k^{1/n} = \int_{-1}^{1} |f^{(n+1)}(t)|^{1/n}\,dt = \lambda\,.$$

Fix $-1 \le \alpha \le \beta \le 1$ and let τ be any weak limit point of the sequence of unit measures (τ_k). Again using the Riemann integrability of $|f^{(n+1)}|^{1/n}$, we have

$$\tau([\alpha, \beta])\lambda = \lim_{k\to\infty} (k+1)\tau_k([\alpha, \beta])cE_k^{1/n} \doteq \int_{\alpha}^{\beta} |f^{(n+1)}(t)|^{1/n}\,dt\,.$$

This proves the theorem. □

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